Find the functional dependency relationship between applications

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Abstract.

An algorithmic method is proposed to solve functional dependency relationships between applications. To do so, a simple theorem is stated and 3 examples are provided, including the solution to show the effectiveness of the method.

I am a mechanical industrial engineer but I am still interested in mathematics. Two months ago I reread a book I had studied and saw a problem where the author established the functional dependency relationship between applications by simple observation but not by mathematical analysis. I try to put this job in https://arxiv.org/ but I have not endorsers who observe my work because I am not dedicated to research or teaching.

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Theorem 1. Method to find the functional dependency relationship in the case of dependent applications $m = n$.

Let the family $F = \{f_1, f_2, ... f_m\}$ of applications of $\mathbb{R}^n$ in $\mathbb{R}$ being:

$$\mathrm{F} = \begin{cases} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_m(x_1, x_2, \cdots, x_n) \end{cases}$$

For the case $n = m$, we have A an open set of $\mathbb{R}^n$ and the family F of class one applications of A in $\mathbb{R}$. Then,

$$\mathrm{F} = \begin{cases} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{cases}$$

It is known that the necessary and sufficient condition for the family F to be functionally dependent in A is that the determinant of its Jacobian matrix is identically zero in A. Since $m = n$, the matrix is square and the Jacobian determinant can be calculated. Then, if the functional relation for the family F to be functionally dependent is:

$$F^0 = c_1 f_1^{a_1} + c_2 f_2^{a_2} + \cdots + c_n f_n^{a_n} = 0 \qquad (1)$$

where the values $c_1 = c_2 = \cdots = c_n \neq 0$, are numerical coefficients and the values $a_1 = a_2 = \cdots = a_n \neq 0$, are numerical exponents of the applications and we can start, to obtain the calculation of equation 1, from the following expression:

$$dF^0 = b_1 df_1 + b_2 df_2 + \cdots + b_n df_n = 0 \qquad (2)$$

where $b_1, b_2, \dots, b_n$ are terms (numerical coefficients that can be multiplied, some of them, by any of the applications $f_1, f_2, \dots, f_n$) after solving the following system of equations,

$$
\begin{cases}
b_1 \dfrac{\partial f_1}{\partial x_1} + b_2 \dfrac{\partial f_1}{\partial x_2} + \cdots + b_n \dfrac{\partial f_1}{\partial x_n} = 0 \\
b_1 \dfrac{\partial f_2}{\partial x_1} + b_2 \dfrac{\partial f_2}{\partial x_2} + \cdots + b_n \dfrac{\partial f_2}{\partial x_n} = 0 \\
\vdots \\
b_1 \dfrac{\partial f_n}{\partial x_1} + b_2 \dfrac{\partial f_n}{\partial x_2} + \cdots + b_n \dfrac{\partial f_n}{\partial x_n} = 0
\end{cases}
$$

Technically, equation 2 could be expressed as follows,

$$c_1 a_1 {f_1}^{a_1-1} df_1 + c_2 a_2 {f_2}^{a_2-1} df_2 + \cdots + c_n a_n {f_n}^{a_n-1} df_n = 0 \qquad (2a)$$

It turns out that when identifying terms,

$$b_1 = c_1 a_1 {f_1}^{a_1-1},\ b_2 = c_2 a_2 {f_2}^{a_2-1},\ \dots,\ b_n = c_n a_n {f_n}^{a_n-1}$$

However, the system generated to solve equation 2 is sufficient for solving the functional dependency relationship between the applications. Therefore, equation 2a is ignored.

The reason why the exponential coefficients $a_1 = a_2 = \cdots = a_n$ are ignored is simple: when finding the differential of each summand these terms are absorbed into the $b_1, b_2, \dots, b_n$ terms.

DEMONSTRATION.

Formally we need to perform the following operation, product of the row vector of coefficients by the column vector of the applications to arrive at the functional dependency relationship:

$$F^0(x_1, x_2, \cdots, x_n) = (c_1 \quad c_2 \quad \cdots \quad c_n)\begin{pmatrix} {f_1}^{a_1} \\ {f_2}^{a_2} \\ \vdots \\ {f_n}^{a_n} \end{pmatrix} = (0)$$

$$F^0 = c_1{f_1}^{a_1} + c_2{f_2}^{a_2} + \cdots + c_n{f_n}^{a_n} = 0$$

where we have arrived at equation 1, easily.

Since F^0 is of class one, it admits the first total derivative. It is known that the derivative is the inverse function of the integration. Therefore, knowing equation 2,

$$dF^0 = b_1 df_1 + b_2 df_2 + \cdots + b_n df_n = 0$$

equation 1 can be easily found by integration, although equation 2 must first be constructed based on obtaining the terms $b_1, b_2, \ldots b_n$, instead of trying to find them directly using the coefficients $c_1, c_2, \ldots c_n$ of equation 1.

It is known by hypothesis that the family F is functionally dependent if it is a $m\,x\,n$ applications where $m = n$ if its Jacobian determinant is identically zero

The Jacobian matrix of the family F is,

$$J(x_1, x_2, \cdots, x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

And its Jacobian determinant is identically null, a necessary and sufficient condition for the family of applications to be functionally dependent. This is already proven in another theorem, so,

$$|J(x_1, x_2, \cdots, x_n)| = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} = \Delta = 0$$

If $|\mathrm{J}(x_1, x_2, \cdots, x_n)| = \Delta = 0 \Leftrightarrow \mathrm{F}$ is a family of functionally dependent applications

Because the F family is class one, it admits the first derivative, as is obvious and has already been mentioned previously. Therefore, the total derivative of the family can be obtained.

$$dF = \begin{cases} df_1(x_1, x_2, \cdots, x_n) \\ df_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ df_n(x_1, x_2, \cdots, x_n) \end{cases}$$

Let us express the total derivatives of the applications in terms of their partial derivatives,

$$\begin{aligned} df_1(x_1, x_2, \cdots, x_n) = df_1 &= \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \cdots + \frac{\partial f_1}{\partial x_n} dx_n \\ df_2(x_1, x_2, \cdots, x_n) = df_2 &= \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 + \cdots + \frac{\partial f_2}{\partial x_n} dx_n \\ &\vdots \\ df_n(x_1, x_2, \cdots, x_n) = df_n &= \frac{\partial f_n}{\partial x_1} dx_1 + \frac{\partial f_n}{\partial x_2} dx_2 + \cdots + \frac{\partial f_n}{\partial x_n} dx_n \end{aligned} \tag{3}$$

In equation 2, we include the partial derivatives of equation 3,

$$\begin{aligned} &b_1 \left(\frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \cdots + \frac{\partial f_1}{\partial x_n} dx_n \right) + \\ &b_2 \left(\frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 + \cdots + \frac{\partial f_2}{\partial x_n} dx_n \right) + \\ &\qquad \vdots \qquad\qquad\qquad\qquad + \\ &b_\mathrm{n} \left(\frac{\partial f_\mathrm{n}}{\partial x_1} dx_1 + \frac{\partial f_\mathrm{n}}{\partial x_2} dx_2 + \cdots + \frac{\partial f_\mathrm{n}}{\partial x_n} dx_n \right) = 0 \end{aligned} \tag{4}$$

It is regrouped in equation 4.

$$\begin{aligned} &\left(b_1 \frac{\partial f_1}{\partial x_1} + b_2 \frac{\partial f_2}{\partial x_1} + \cdots + b_\mathrm{n} \frac{\partial f_\mathrm{n}}{\partial x_1} \right) dx_1 + \\ &\left(b_1 \frac{\partial f_1}{\partial x_2} + b_2 \frac{\partial f_2}{\partial x_2} + \cdots + b_\mathrm{n} \frac{\partial f_\mathrm{n}}{\partial x_2} \right) dx_2 + \\ &\qquad \vdots \qquad\qquad\qquad\qquad + \\ &\left(b_1 \frac{\partial f_1}{\partial x_\mathrm{n}} + b_2 \frac{\partial f_2}{\partial x_\mathrm{n}} + \cdots + b_\mathrm{n} \frac{\partial f_\mathrm{n}}{\partial x_\mathrm{n}} \right) dx_\mathrm{n} = 0 \end{aligned} \tag{5}$$

For equation 5 to represent a functionally dependent family, it must simultaneously happen that,

a) $dx_1 = dx_2 = \cdots = dx_\mathrm{n} \neq 0$

b)

$$\begin{aligned} b_1 \frac{\partial f_1}{\partial x_1} + b_2 \frac{\partial f_2}{\partial x_1} + \cdots + b_\mathrm{n} \frac{\partial f_\mathrm{n}}{\partial x_1} &= 0 \\ b_1 \frac{\partial f_1}{\partial x_2} + b_2 \frac{\partial f_2}{\partial x_2} + \cdots + b_\mathrm{n} \frac{\partial f_\mathrm{n}}{\partial x_2} &= 0 \\ &\vdots \\ b_1 \frac{\partial f_1}{\partial x_\mathrm{n}} + b_2 \frac{\partial f_2}{\partial x_\mathrm{n}} + \cdots + b_\mathrm{n} \frac{\partial f_\mathrm{n}}{\partial x_\mathrm{n}} &= 0 \end{aligned} \tag{6}$$

Equation 6 represents a system of n equations with n unknowns. Its expanded matrix is represented including the null coefficients.

$$\left(\begin{array}{cccc|c} b_1\frac{\partial f_1}{\partial x_1} & b_2\frac{\partial f_2}{\partial x_1} & \cdots & b_\mathrm{n}\frac{\partial f_\mathrm{n}}{\partial x_1} & 0 \\ b_1\frac{\partial f_1}{\partial x_2} & b_2\frac{\partial f_2}{\partial x_2} & \cdots & b_\mathrm{n}\frac{\partial f_\mathrm{n}}{\partial x_2} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_1\frac{\partial f_1}{\partial x_\mathrm{n}} & b_2\frac{\partial f_2}{\partial x_\mathrm{n}} & \cdots & b_\mathrm{n}\frac{\partial f_\mathrm{n}}{\partial x_\mathrm{n}} & 0 \end{array}\right) \tag{7}$$

The system could be solved by obtaining an upper triangular matrix using the Gauss method and using the back-up or backward substitution technique to find the terms $b_1, b_2, \dots, b_\mathrm{n}$ that contain numerical coefficients or numerical coefficients multiplying some of the applications $f_1, f_2, \dots, f_n$. It can also be solved directly by manipulating the equations of the system to find the terms.

When finding the terms $b_1, b_2, \dots, b_\mathrm{n}$ we must look for the equalities that arise from the variables $x_1, x_2, \dots, x_\mathrm{n}$ and their relations with the applications $f_1, f_2, \dots, f_\mathrm{n}$ so that the terms $b_1, b_2, \dots, b_\mathrm{n}$ only have numerical coefficients by multiplying some of the applications instead of the variables.

Finding the terms $b_1, b_2, \dots, b_\mathrm{n}$ for equation 2,

$$dF^0 = b_1 df_1 + b_2 df_2 + \cdots + b_n df_n = 0$$

the integration process of F^0 is carried out, for each of the summands on its variable, equation 1 is reached,

$$F^0 = c_1 {f_1}^{a_1} + c_2 {f_2}^{a_2} + \cdots + c_n {f_n}^{a_n} = 0$$

In this way, the problem of finding the linear dependency relationship between $n\,x\,n$ applications has been solved.

c.q.d.

Solved practical examples of the previous theorem.

Exercise 1.

Let the family $\{\boldsymbol{f_1}, \boldsymbol{f_2}, \boldsymbol{f_3}\}$ of applications of $\mathbb{R}^3$ in $\mathbb{R}$ be defined by:

$$\boldsymbol{f_1(u,v,w) = x = u^2 + v^2 + w^2}$$
$$\boldsymbol{f_2(u,v,w) = y = u + v + w}$$
$$\boldsymbol{f_3(u,v,w) = z = uv + vw + wu}$$

1) Show that the family is functionally dependent in all $\mathbb{R}^3$.
2) Find an appropriate method to detail the functional relationship. You will have to find a functional relationship expression of this type,

$$\boldsymbol{F(x,y,z) = mx^i + ny^j + pz^k = 0}$$

Note that if the previous expression were $\boldsymbol{F(x,y,z) \neq 0}$ the family would be functionally independent.

1)

Let be the matrix associated with the family f formed by the given applications.

$$f(u,v,w) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} u^2 + v^2 + w^2 \\ u + v + w \\ uv + vw + wu \end{pmatrix}$$

We can obtain the determinant of its Jacobian matrix. If it is null, there is functional dependence.

$$\frac{D(x,y,x)}{D(u,v,w)} = \det[f'(u,v,w)] = \det[J(u,v,w)] = \begin{vmatrix} 2u & 2v & 2w \\ 1 & 1 & 1 \\ v+w & u+w & u+v \end{vmatrix} = \Delta$$

$$\Delta = 2u(u+v) + 2v(v+w) + 2w(u+w) - 2w(v+w) - 2u(u+w) - 2v(u+v) =$$
$$2u^2 + 2uv + 2v^2 + 2vw + 2wu + 2w^2 - 2vw - 2w^2 - 2u^2 - 2wu - 2uv - 2v^2 = 0$$

Therefore, since the determinant is null, there is functional dependence and the family is functionally dependent $\forall\, (u,v,w) \in \mathbb{R}$.

2) We tried a resolution method.

We have identified each application with a variable, such that,

$$f_1 = x$$
$$f_2 = y$$
$$f_3 = z$$

We started with the equation,

$$F(x,y,z) = mx^i + ny^j + pz^k$$

Developing the equation,

$$F(x,y,z) = m(u^2 + v^2 + w^2)^i + n(u + v + w)^j + p(uv + vw + wu)^k = 0$$

We cannot obtain anything by expanding the equation. The equation in this form is analytically unsolvable.

Theorem 1 will be used to develop the systematic method for finding the dependency relationship between the different applications.

That is, we have to find the equation,

$$F(x,y,z) = mx^i + ny^j + pz^k = 0 \qquad (8)$$

which relates the functional dependency between the applications of the F family, using unknown terms.

We just need to find the coefficients m, n, p.

We will explain the procedure in a practical way for this exercise.

Let the expression be,

$$adx + bdy + cdz = 0 \qquad (9)$$

differential equation derived from equation 26 with the terms a, b y c.

Please note that in the resolution of the exercise, another nomenclature is being used in the variables, applications and terms that are more appropriate for the resolution of practical problems. The following equivalences have been established,

$$(f_1, f_2, f_3) = (x, y, z)$$
$$(c_1, c_2, c_3) = (m, n, p)$$
$$(b_1, b_2, b_3) = (a, b, c)$$

The procedure involves finding the previous terms to integrate the differential equation 9, obtaining the equation of functional dependence between the applications, arriving at equation 8.

The differential equation of each of the applications has been taken and multiplied by each of the terms.

x, y, z are defined according to the applications as follows,

$$x = f_1(u, v, w) = u^2 + v^2 + w^2$$
$$y = f_2(u, v, w) = u + v + w$$
$$z = f_3(u, v, w) = uv + vw + wu$$

$$J(u, v, w) = \begin{pmatrix} 2u & 2v & 2w \\ 1 & 1 & 1 \\ v + w & u + w & u + v \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

We developed the expression,

$$adx + bdy + cdz = 0$$

knowing that the total differentials dx, dy, dz are,

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$$
$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial w} dw$$
$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw$$

So, we have,

$$a\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw\right) + b\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial w} dw\right) + c\left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw\right) = 0$$

$$\left(a\frac{\partial x}{\partial u} + b\frac{\partial y}{\partial u} + c\frac{\partial z}{\partial u}\right) du + \left(a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial v}\right) dv + \left(a\frac{\partial x}{\partial w} + b\frac{\partial y}{\partial w} + c\frac{\partial z}{\partial w}\right) dw = 0$$

Substituting the values of the Jacobian matrix, in the above expression,

$$[2au + b + c(v + w)]\, du + [2av + b + c(u + w)]dv + [2aw + b + c(u + v)]dw = 0$$

Considering $du \neq 0, dv \neq 0, dw \neq 0$, the values in the brackets must necessarily equal zero.

We examine the values in the brackets and obtain a system of 3 equations to clear the coefficients a, b y c.

$$2au + b + c(v + w) = 0 \quad (10)$$
$$2av + b + c(u + w) = 0 \quad (11)$$
$$2aw + b + c(u + v) = 0 \quad (12)$$

We carry out the following operations,

- equation 10 – equation 11:

$$2a(u - v) + c(v - u) = 0 \to (2a - c)(u - v) = 0 \to \begin{cases} u = v \ (no\ interest) \\ 2a = c \end{cases}$$

Then,

$$a = \frac{c}{2} \quad (13)$$

We carry the result of equation 13 to equation 10,

$$2\left(\frac{c}{2}\right)v + b + c(u + w) = 0 \to b = \ -c(u + v + w) \to b = -cy = -cf_2$$

$$b = -cy \quad (14)$$

Therefore, we consider,

$$c \ = \ 2 \quad (15)$$

And in this way a is not a fractional number to make the solution simpler,
because it is the lowest value that can be taken as a positive integer and then it results,

$$a = 1, \qquad c = 2, \qquad b = -2y \quad (16)$$

which is the solution of the terms to the differential equation 2. Note that both a and c are numerical coefficients but b is a term.

We carry these values into Eq. (2),

$$adx + bdy + cdz = 0 \to dx - 2ydy + 2dz = 0$$

Integrating the differential equation, we get,

$$x - y^2 + 2z = K = F(x, y, z)$$

But we know that K, the integration constant equals zero, since $F(x, y, z) = 0$ for there to be functional dependence between the applications, then,

$$x - y^2 + 2z = 0 \Leftrightarrow f_1 - (f_2)^2 + 2f_3 = 0 \tag{17}$$

Testing,

$$(u^2 + v^2 + w^2) - (u + v + w)^2 + 2(uv + vw + wu) = 0$$

It is known that,

$$(u + v + w)^2 = u^2 + v^2 + w^2 + 2uv + 2vw + 2wu$$

Developing,

$$u^2 + v^2 + w^2 - (u^2 + v^2 + w^2 + 2uv + 2vw + 2wu) + 2(uv + vw + wu) =$$
$$u^2 + v^2 + w^2 - u^2 - v^2 - w^2 - 2uv - 2vw - 2wu + 2uv + 2vw + 2wu = 0$$

It is clear that the solution is correct so it can be seen that the theorem is feasible to find the functional dependency relationship between applications.

Exercise 2.

Let the family $\{\boldsymbol{f_1}, \boldsymbol{f_2}, \boldsymbol{f_3}\}$ of applications of $\mathbb{R}^3$ in $\mathbb{R}$ be defined by:

$$\boldsymbol{f_1(u,v,w) = x = u^2 + v^2 + w^2}$$
$$\boldsymbol{f_2(u,v,w) = y = u + v + w}$$
$$\boldsymbol{f_3(u,v,w) = z = u^2 + v^2 + w^2 + 6uv + 6vw + 6uw}$$

1) Show that the family is functionally dependent in all $\mathbb{R}^3$.
2) Find an appropriate method to detail the functional relationship. You will have to find a functional relationship expression of this type,

$$\boldsymbol{F(x,y,z) = mx^i + ny^j + pz^k = 0}$$

Note that if the previous expression were $\boldsymbol{F(x,y,z) \neq 0}$ the family would be functionally independent.

1)

Let be the matrix associated with the family f formed by the given applications.

$$f(u,v,w) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} u^2 + v^2 + w^2 \\ u + v + w \\ u^2 + v^2 + w^2 + 6uv + 6vw + 6uw \end{pmatrix}$$

We can obtain the determinant of its Jacobian matrix. If null, functional dependency exists.

$$\det[J(u,v,w)] = \begin{vmatrix} 2u & 2v & 2w \\ 1 & 1 & 1 \\ 2u + 6(v+w) & 2v + 6(u+w) & 2w + 6(v+u) \end{vmatrix} = \Delta$$

$$\Delta = 2u[2w + 6(v+u)] + 2v[2u + 6(v+w)] + 2w[2v + 6(u+w)] - 2w[2u + 6(v + w)] - 2v[2w + 6(v+u)] - 2u[2v + 6(u+w)] =$$
$$2u[2w + 6v + 6u - 2v - 6u - 6w] + 2v[2u + 6v + 6w - 2w - 6v - 6u]$$
$$+ 2w[2v + 6u + 6w - 2u - 6v - 6w]$$
$$= 2u[4v - 4w] + 2v[4w - 4u] + 2w[4u - 4v]$$
$$= 8uv - 8uw + 8vw - 8uv + 8uw - 8vw = 0$$

Therefore, since it is the null determinant, there is functional dependence and the family is functionally dependent $\forall\ (u,v,w) \in\ \mathbb{R}$.

2)

The theorem that develops a systematic method to find the dependency relationship between the different applications will be used again.

We must find the relationship,

$$F(x,y,z) = mx^i + ny^j + pz^k = 0 \tag{18}$$

that relates the functional dependency between the applications of the F family, using unknown terms.

We just need to find the coefficients m, n, p.

We will explain the procedure again in a practical way for this new exercise.

Be the expression,

$$adx + bdy + cdz = 0 \tag{19}$$

differential equation derived from equation 18 with the terms a, b and c.

The following equivalences are established again:

$$(f_1, f_2, f_3) = (x, y, z)$$
$$(c_1, c_2, c_3) = (m, n, p)$$
$$(b_1, b_2, b_3) = (a, b, c)$$

The procedure involves finding the previous terms to integrate differential equation 19, obtaining the equation of functional dependence between the applications, arriving at equation 18.

The differential equation of each of the applications has been taken and multiplied by each of the terms.

x, y, z are defined according to the applications as follows,

$$x = f_1(u,v,w) = u^2 + v^2 + w^2$$
$$y = f_2(u,v,w) = u + v + w$$
$$z = f_3(u,v,w) = u^2 + v^2 + w^2 + 6uv + 6vw + 6uw$$

$$J(u,v,w)=\begin{pmatrix}2u & 2v & 2w\\ 1 & 1 & 1\\ 2u+6(v+w) & 2v+6(u+w) & 2w+6(v+u)\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{pmatrix}$$

We develop the expression,

$$adx+bdy+cdz=0$$

knowing that the total differentials dx, dy, dz are:

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw$$
$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv+\frac{\partial y}{\partial w}dw$$
$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv+\frac{\partial z}{\partial w}dw$$

So, we have,

$$a\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw\right)+b\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv+\frac{\partial y}{\partial w}dw\right)$$
$$+c\left(\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv+\frac{\partial z}{\partial w}dw\right)=0$$

$$\left(a\frac{\partial x}{\partial u}+b\frac{\partial y}{\partial u}+c\frac{\partial z}{\partial u}\right)du+\left(a\frac{\partial x}{\partial v}+b\frac{\partial y}{\partial v}+c\frac{\partial z}{\partial v}\right)dv+\left(a\frac{\partial x}{\partial w}+b\frac{\partial y}{\partial w}+c\frac{\partial z}{\partial w}\right)dw=0$$

Substituting the values of the Jacobian matrix, in the previous expression,

$$[2au+b+c(2u+6v+6w)]\,du+[2av+b+c(2v+6u+6w)]dv$$
$$+[2aw+b+c(2w+6u+6v)]dw=0$$

Since $du\neq 0, dv\neq 0, dz\neq 0$, the values of the square brackets must necessarily be zero.

We examine the values of the brackets and obtain a system of 3 equations to solve for the coefficients a, b y c.

$$2au+b+c(2u+6v+6w)=0 \qquad (20)$$
$$2av+b+c(2v+6u+6w)=0 \qquad (21)$$
$$2aw+b+c(2w+6u+6v)=0 \qquad (22)$$

Equations 20 - 21:

$$2a(u-v)+c[2u+6v+6w-2v-6u-6w)]=0\Rightarrow$$
$$2a(u-v)+c[4v-4u)]=0\Rightarrow$$
$$2a(u-v)-4c[u-v)]=0\Rightarrow\begin{cases}u=v\\ a=2c\end{cases}$$

We introduce the above calculation into Eq. 20, by entering the value obtained from a,

$$4cv + b + c(8v + 6w) = 0 \Longrightarrow b + 12cv + 6cw = 0 \Longrightarrow b = -6c(w + 2v)$$

We know that,

$$y = u + v + w$$

So,

$$w + 2v = y - u - v + 2v = y - u + v$$

But before we get to that,

$$u = v$$

obtained from the previous calculation.

Therefore,

$$w + 2v = y$$

And then,

$$b = -6cy$$

Setting $c = 1$, we have,

$$a = 2$$
$$b = -6y$$

Putting the terms in the differential equation 19,

$$2dx - 6ydy + dz = 0$$

Integrating the above equation, we get:

$$2x - 3y^2 + z = K = F$$

With $K = 0$ because there is a linear dependency relationship, we have,

$$2x - 3y^2 + z = 0 \Leftrightarrow 2f_1 - 3f_2^2 + f_3 = 0$$

It is not proven that this relationship is true as in the previous exercise.

Exercise 3.

Let the family $\{\boldsymbol{f_1}, \boldsymbol{f_2}, \boldsymbol{f_3}\}$ of applications of $\mathbb{R}^3$ in $\mathbb{R}$ be defined by:

$$\boldsymbol{f_1(u,v,w) = x = u}$$
$$\boldsymbol{f_2(u,v,w) = y = v + w}$$
$$\boldsymbol{f_3(u,v,w) = z = -u^3 + 2v^2 + 2w^2 + 4vw}$$

1) Show that the family is functionally dependent in all $\mathbb{R}^3$.
2) Find an appropriate method to detail the functional relationship. You will have to find a functional relationship expression of this type,

$$\boldsymbol{F(x,y,z) = mx^i + ny^j + pz^k = 0}$$

Note that if the previous expression were $\boldsymbol{F(x,y,z) \neq 0}$ the family would be functionally independent.

1)

Let be the matrix associated with the family f formed by the given applications.

$$f(u,v,w) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} u \\ v + w \\ -u^3 + 2v^2 + 2w^2 + 4vw \end{pmatrix}$$

We can obtain the determinant of its Jacobian matrix. If it is null, there is functional dependence.

$$\det[J(u,v,w)] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3u^2 & 4v + 4w & 4v + 4w \end{vmatrix} = \Delta$$

$$\Delta = 4v + 4w + 0 + 0 - 0 - 4v - 4w = 0$$

Therefore, since the determinant is null, there is functional dependence and the family is functionally dependent $\forall\ (u,v,w) \in\ \mathbb{R}$.

The theorem that develops a systematic method to find the dependency relationship between different applications will be used again.

We must find the relationship,

$$F(x, y, z) = mx^i + ny^j + pz^k = 0 \tag{23}$$

which relates the functional dependency between the applications of the F family, using unknown terms.

We just need to find the coefficients m, n, p.

We will explain the procedure again in a practical way for this new exercise.

Let the expression be,

$$adx + bdy + cdz = 0 \tag{24}$$

Differential equation derived from equation 23 with terms a, b and c.

The following equivalences are established again,

$$(f_1, f_2, f_3) = (x, y, z)$$
$$(c_1, c_2, c_3) = (m, n, p)$$
$$(b_1, b_2, b_3) = (a, b, c)$$

The procedure involves finding the previous terms to integrate the differential equation 24, obtaining the equation of functional dependence between the applications, arriving at equation 23.

The differential equation of each of the applications has been taken and multiplied by each of the terms.

x, y, z are defined, according to the applications as follows,

$$x = f_1(u, v, w) = u$$
$$y = f_2(u, v, w) = v + w$$
$$z = f_3(u, v, w) = -u^3 + 2v^2 + 2w^2 + 4vw$$

$$J(u,v,w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3u^2 & 4v+4w & 4v+4w \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

We developed the expression,

$$adx + bdy + cdz = 0$$

knowing that the total differentials dx, dy, dz are:

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw$$
$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw$$
$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

So, we have,

$$a\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right) + b\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw\right) + c\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw\right) = 0$$

$$\left(a\frac{\partial x}{\partial u} + b\frac{\partial y}{\partial u} + c\frac{\partial z}{\partial u}\right)du + \left(a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial v}\right)dv + \left(a\frac{\partial x}{\partial w} + b\frac{\partial y}{\partial w} + c\frac{\partial z}{\partial w}\right)dw = 0$$

Substituting the values of the Jacobian matrix, in the above expression,

$$[a + 0 + c(-3u^2)]\,du + [0 + b + c(4v + 4w)]dv + [0 + b + c(4v + 4w)]dw = 0$$

Since $du \neq 0, dv \neq 0, dz \neq 0,$ the values in the brackets must necessarily equal zero.

We examine the values in the brackets and obtain a system of 3 equations to clear the coefficients a, b and c.

$$a + c(-3u^2) = 0 \quad (25)$$
$$b + c(4v + 4w) = 0 \quad (26)$$
$$b + c(4v + 4w) = 0 \quad (27)$$

From Eq. 25:

$$a = 3cu^2 \Longrightarrow a = 3cx^2$$

From Eq. 26 o 27:

$$b = -4c(v + w) \Longrightarrow b = -4cy$$

Taking the value $c = 1$, we have,

$$a = 3x^2$$
$$b = -4y$$

Putting the terms in the differential equation 24,

$$3x^2dx - 4ydy + dz = 0$$

Integrating the above equation, we get:

$$x^3 - 2y^2 + z = K = F$$

With $K = 0$ because there is a linear dependency relationship, we have,

$$x^3 - 2y^2 + z = 0 \Leftrightarrow f_1^3 - 2f_2^2 + f_3 = 0$$

It is not proven that this relationship is true as in the first exercise.

References.

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